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Inverse Spectral Problems for Singular Sturm-Liouville Operators

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ABSTRACT

In this work, inverse problems for singular Sturm-Liouville operators at the finite interval are studied. In this study, we show the spectral characteristics and prove the uniqueness theorem for the solution of the inverse problem. Finally, we give an applied example and use the numerical technique to obtain the approximate solution of the problem.

Keywords: Sturm-Liouville operator, Singularity, Weyl function.

1. Introduction

Inverse problems for differential equations with singularities inside an interval are important in mathematics and its applications. A wide class of differential equations with turning points can be reduced to equations with singularities (see Neamaty and Mosazadeh (2011)). Indefinite boundary value problems with a continuous weight function that has several zeros were studied in Eberhard et al. (2001). Furthermore, a wide class of differential equations with Bessel-type singularities and perturbations can be reduced to differential equations having turning points. We can see singular differential equations in various problems of natural sciences like geophysical models of the earth's crust and an embankment (see for example Anderssen (1997)).

In this paper, we consider the indefinite Sturm-Liouville problem

$$-y'' + \left(\frac{\frac{3}{4}}{(x-H)^2} + \frac{\frac{-1}{2H}}{x-H} + q(x) \right) y = \lambda y, \quad h < x < H, \tag{1}$$

$$U(y) := y'(h) - \beta y(h) = 0, \qquad V(y) := y(H) = 0, \tag{2}$$

for $\beta = \frac{H}{2h(H-h)}$. Here the real function $q(x)$ belongs to $L_2(h,H)$ and $h,\ H$ are positive real numbers. Also $\lambda = \rho^2$ is a spectral parameter.

This paper proposes and tests an extension of the technique studied in Amirov and Topsakal (2008) for obtaining the solution of the direct Sturm-Liouville problem. Similarly singular Sturm-Liouville problems of the first and second order have been studied by some researchers (see Koyunbakan (2009), Koyunbakan and Bulut (2005), Neamaty and Mosazadeh (2011), Topsakal and Amirov (2010) and Yurko (1997)).

For example, in the article Topsakal and Amirov (2010), the authors considered the following Sturm-Liouville problem with a Coulomb potential

$$-y'' + \left(\frac{C}{x} + q(x) \right) y = \lambda y, \quad 0 < x \le \pi,$$

$$y(0) = y(\pi) = 0,$$

along with the jump condition

$$y^{(m)}(d+0) = a^{(-1)^m} y^{(m)}(d-0), \quad m = 0, 1,$$

wherein the real function $q(x)$ belongs to $L_2(0,\pi)$ and $a \in R^+, a \neq 1, d \in (\frac{\pi}{2}, \pi)$.

In this paper, by taking the changes of variables $y_1(x) = y(x)$ and $y_2(x) = y'(x) - u(x)y(x)$ for $u(x) = Clnx$, the authors transform the equation to a system and then get a solution to it. In the article Yurko (1997), singular Sturm-Liouville problems of Bessel-type,

$$-y'' + \left(\frac{v_0}{(x-x_0)^2} + q(x) \right) y = \lambda y, \quad 0 \leq x \leq T,$$

$$y(0) = y(T) = 0.$$

have been investigated. It has been assumed that v_0 is a complex number and $x_0 \in (0,T)$ for a real number T.

Direct and inverse problems for the classical Sturm-Liouville operators without singularities have been studied in Eberhard et al. (1958) and Naimark (1967). The inverse problem is a problem which studies a method of the reconstruction of operators by some data. This problem was first surveyed by Ambartsumyan in 1929 (see Ambartsumyan (1929)). Since 1946, a number of authors like Borg, Levinson and Levitan has studied the inverse problem in different cases (see Borg (1945), Levinson (1949) and Levitan (1978)). Later the inverse problems having different conditions were surveyed in Freiling and Yurko (2001), Levitan (1987), Naimark (1967), Neamaty and Khalili (2015), Topsakal and Amirov (2010) and Yurko (1997).

Inverse problems for differential operators without singularities have been studied in McLaughlin (1986). The presence of the singularity in these problems makes essential difficulties in the investigation of the inverse problem. Inverse problems for the Sturm-Liouville equation with singularities of a second order have been studied by authors like Koyunbakan and Yurko (see Koyunbakan (2005) and Yurko (1997)). In other works, singular Sturm-Liouville operators of a first order have been studied by different techniques in obtaining the solution of the direct problem (see Amirov and Topsakal (2008) and Topsakal and Amirov (2010)). The novelty of this paper is to study the singular Sturm-Liouville operator of the first and second order together. In this paper, we will apply the method used in Amirov and Topsakal (2008) to obtain the solution of the direct problem.

Note that similar problems for the Sturm-Liouville operator have been studied in Neamaty and Mosazadeh (2011), Topsakal and Amirov (2010) and Yurko (1997). Here we investigate the inverse problem of recovering equation (1) under the boundary conditions (2) from the given Weyl function. We first present properties of the spectrum. Then we introduce the so-called Weyl function which is a generalization of the notion of the Weyl function for the classical Sturm-Liouville operators.

In this work, to prove the uniqueness theorem, the Weyl function for the considered operator has been defined. In Sec. 2, we determine the asymptotic form of the solutions of (1) and establish the Weyl function. In Sec. 3, the uniqueness theorem for the solution of the inverse problem has been proved. Finally, in Sec. 4, we give an example about the shaping of the singular Sturm-Liouville equation in a natural phenomenon and take a numerical technique for obtaining it's solution. We note that throughout the whole paper, we will call the boundary value problem (1)-(2) the so-called BVP(L).

2. Spectral Data Weyl function

At first taking the method presented in Amirov and Topsakal (2008), we get the solution of the Eq. (1). For this purpose, we consider $y_1(x) = y(x)$ and $y_2(x) = (\Gamma y)(x) = y'(x) - u(x)y(x)$ for $u(x) = \frac{3}{4}(H-x) - \frac{1}{2H}ln(H-x)$, and we have the equation (1) to the form

$$\ell(y) := -\big((\Gamma y)(x)\big)' - u(x)(\Gamma y)(x) - u^2(x)y(x) + q(x)y(x) = \lambda y(x). \tag{3}$$

So Eq. (3) can be reduced to

$$\begin{cases} uy_1 + y_2 = y_1', \\ \\ \left(-\lambda - u^2 + q(x)\right) y_1 - uy_2 = y_2', \end{cases} \tag{4}$$

with the boundary conditions

$$U(y_1) := y_1'(h) - \beta y_1(h) = 0, \qquad V(y_1) := y_1(H) = 0. \tag{5}$$

The matrix form of the system (4) is writing in the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -\lambda - u^2 + q & -u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

By regarding to Naimark (1967), the system (4) has only one solution satisfying the initial conditions $y_1(\xi) = \nu_1$ and $y_2(\xi) = \nu_2$ for each $\xi \in (h, H)$, $\nu = (\nu_1, \nu_2)^T \in C^2$, especially $y_1(h) = 1$ and $y_2(h) = i\rho$.

Definition 1 (Amirov and Topsakal (2008)). The first component of the solution of (4) which satisfies the initial conditions $y_1(\xi) = \nu_1$ and $y_2(\xi) = \nu_2$ is called the solution of (1) which satisfies the same initial conditions.

Let us denote a solution of the system (4) in the case $q(x) = 0$ and $C = 0$, i.e.,

$$\begin{cases} y_1' - y_2 = 0, \\ y_2' + \lambda y_1 = 0, \end{cases} \tag{6}$$

for the differential equation

$$y'' + \lambda y = 0, \tag{7}$$

by $\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}(x, \lambda)$ together with the initial condition $\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}(h, \lambda) = \begin{pmatrix} 1 \\ i\rho \end{pmatrix}$.

The solution of the system (6) satisfying the initial condition $\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}(h, \lambda) = \begin{pmatrix} 1 \\ i\rho \end{pmatrix}$ can be written as

$$\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}(x, \lambda) = \begin{pmatrix} e^{i\rho(x-h)} \\ i\rho e^{i\rho(x-h)} \end{pmatrix}. \tag{8}$$

By the successive approximations method (see Freiling and Yurko (2001) and Marchenko (1986)), from Amirov and Topsakal (2008) we have the following theorem.

Theorem 1. The system (4) together with the initial condition $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(h, \lambda) =$

$\begin{pmatrix} 1 \\ i\rho \end{pmatrix}$ has the following integral solution:

$$y_1(x,\lambda) = y_{01}(x,\lambda) + \int_{-x+h}^{x-h} K_{11}(x,s) y_{01}(s,\lambda) ds, \tag{9}$$

$$\begin{aligned} y_2(x,\lambda) &= y_{02}(x,\lambda) + b(x) y_{01}(x,\lambda) \\ &\quad + \int_{-x+h}^{x-h} K_{21}(x,s) y_{01}(s,\lambda) ds + \int_{-x+h}^{x-h} K_{22}(x,s) y_{02}(s,\lambda) ds, \end{aligned} \tag{10}$$

where

$$\begin{aligned} b(x) &= \frac{-1}{2} \int_0^{x-h} [u^2(s) - q(s)] \exp\left(\frac{-1}{2} \int_s^{x-h} u(r) dr \right) ds, \\ K_{11}(x,x) &= \frac{1}{2} u(x), \\ K_{21}(x,x) &= b'(x) - \frac{1}{2} \int_0^{x-h} [u^2(s) - q(s)] K_{11}(s,s) ds \\ &- \frac{1}{2} \int_0^{x-h} u(s) K_{21}(s,s) ds, \quad K_{22}(x,x) = \frac{-1}{2}(u(x) + 2b(x)). \end{aligned}$$

In the sequel, we consider the properties of the spectrum of L. We also consider the BVP(L_0) for the differential equation (7) together with boundary conditions (2).

Let the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of Eq. (3) satisfying the conditions $\varphi(h,\lambda) = 1$, $(\Gamma\varphi)(h,\lambda) = \beta - \alpha$ for $\alpha = \frac{3}{4}(H-h) - \frac{1}{2H} ln(H-h)$, $\psi(H,\lambda) = 0$ and $(\Gamma\psi)(H,\lambda) = 1$.

It is trivial that $\varphi_0(x,\lambda)$ will be a solution of Eq. (7) with the same conditions. Therefore the solution $\varphi_0(x,\lambda)$ has the form

$$\varphi_0(x,\lambda) = cos\rho(x-h) + \frac{\beta sin\rho(x-h)}{\rho}. \tag{11}$$

Now, we define the characteristic function of L. Denote

$$\Delta(\lambda) = < \psi(x,\lambda), \varphi(x,\lambda) >, \tag{12}$$

where $< y(x), z(x) >:= y(x)(\Gamma z)(x) - (\Gamma y)(x)z(x)$, and is called the Wronskian of the functions $y(x)$ and $z(x)$. Since, by virtue of Liouville's formula, the Wronskian does not depend on x, we can write

$$\Delta(\lambda) = V(\varphi). \tag{13}$$

By using the representation of the function $y(x, \lambda)$ for $\varphi(x, \lambda)$ and $\psi(x, \lambda)$, we have

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \frac{1}{\rho}\int_{-x+h}^{x-h} N(x, s)cos\rho(s-h)ds, \tag{14}$$

$$\psi(x, \lambda) = \psi_0(x, \lambda) + \int_{-H+x}^{H-x} P(x, s)sin\rho(H-h-s)ds, \tag{15}$$

where $N(x, t)$ and $P(x, t)$ are real continuous functions. Thus for sufficiently large ρ, we have

$$\varphi(x, \lambda) = cos\rho(x-h) + O\left(\frac{1}{\rho}exp(|Im\rho|(x-h))\right), \tag{16}$$

$$\psi(x, \lambda) = sin\rho(H-h-x) + O\left(\frac{1}{\rho}exp(|Im\rho|(H-h-x))\right). \tag{17}$$

Now, by taking (2), (13) and (16), the characteristic function for the BVP(L) can be written as

$$\Delta(\rho) = \Delta_0(\rho) + \frac{1}{\rho}\int_{-H+h}^{H-h} N(H, s)cos\rho(s-h)ds. \tag{18}$$

Therefore

$$\Delta(\rho) = \Delta_0(\rho) + O\left(\frac{1}{\rho}exp(|Im\rho|(H-h))\right), \tag{19}$$

where $\Delta_0(\rho) = cos\rho(H-h)$ is a characteristic function for the BVP(L_0).

Theorem 2. The eigenvalues of the BVP(L) satisfy the asymptotic equality

$$\rho_n = \rho_n^0 + O(1), \tag{20}$$

for sufficiently large n. Here ρ_n^0 is the zeros of the function $\Delta_0(\rho)$ and is equal to $\rho_n^0 = \frac{(2n+1)\pi}{2(H-h)}$.

Proof. Denote $G_n = \{\rho; |\rho| = \frac{(2n+1)\pi}{2(H-h)} + \frac{1}{2},\ n = 0, \pm 1, \pm 2, ...\}$.

By taking (18), we can write for a positive constant C,

$$|\Delta(\rho) - \Delta_0(\rho)| \leq \frac{C}{\rho} exp(|Im\rho|(H-h)), \qquad \rho \in G_n$$

(see Marchenko (1986)). Thus

$$\Delta(\rho) = \Delta_0(\rho) + g(\rho), \qquad g(\rho) \leq \frac{C}{\rho} exp(|Im\rho|(H-h)).$$

Since $|\Delta_0(\rho)| > Cexp(|Im\rho|(H-h))$ (see Neamaty and Khalili (2013)), we arrive at $|\Delta_0(\rho)| > |g(\rho)|$ for sufficiently large n. Then according to the Rouche's theorem (see Conway (1995)), the number of zeros of the function $\Delta(\rho)$ and $\Delta_0(\rho)$ is same. Now we can write for sufficiently large n, $\rho_n = \rho_n^0 + \epsilon_n$ wherein ρ_n^0 is the zeros of the function $\Delta_0(\rho)$. It is trivial that $\rho_n^0 = \frac{(2n+1)\pi}{2(H-h)}$. Also it follows from (18) that

$$\Delta(\rho_n) = \Delta_0(\rho_n^0 + \epsilon_n) + \frac{1}{\rho_n} \int_{-H+h}^{H-h} N(H,s) cos\rho_n(s-h)ds.$$

Here $\Delta_0(\rho_n^0 + \epsilon_n) = \Delta_0(\rho_n^0) + \dot{\Delta}_0(\rho_n^0)\epsilon_n$ where $\dot{\Delta} = \frac{d\Delta(\rho)}{2\rho d\rho}$. Therefore

$$\epsilon_n = \frac{1}{\dot{\Delta}_0(\rho_n^0)} \Bigg(\frac{-1}{\rho_n^0} \int_{-H+h}^{H-h} N(H,s) cos\rho_n^0(s-h)ds \\ + O(exp(|Im\rho|(H-h))) \Bigg).$$

Since the function $\dot{\Delta}_0(\rho)$ is type of "cos", we can obtain $\epsilon_n = O(1)$. So the formula (20) is valid. Theorem 2 is proved.

Now, in this part, we introduce the Weyl solution and the Weyl function for the BVP(L). At first, we assume that $C(x,\lambda)$ be a solution of Eq. (3) under the initial conditions $C(h,\lambda) = 0$ and $(\Gamma C)(h,\lambda) = 1$.

We assume that $\phi(x,\lambda)$ be a solution of Eq. (3) considering $U(\phi)=1$ and $V(\phi)=0$. Denote

$$\phi(x,\lambda)=\frac{\psi(x,\lambda)}{\Delta_1(\lambda)}, \tag{21}$$

where $\Delta_1(\rho)=\rho^2\Delta(\lambda)$. We set $M(\lambda):=\phi(h,\lambda)$. The functions $\phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for the BVP(L), respectively. It is obvious that

$$< \phi(x,\lambda),\varphi(x,\lambda) >= -1, \tag{22}$$

$$\phi(x,\lambda)=C(x,\lambda)+M(\lambda)\varphi(x,\lambda). \tag{23}$$

We use the notation

$$\alpha_n := \int_h^H \varphi^2(x,\rho_n)dx. \tag{24}$$

The numbers α_n are called the weight numbers, and the data $S:=\{\rho_n,\alpha_n\}$ are called the spectral data of L.

Theorem 3. The weight numbers of the BVP(L) have the following asymptotic behavior for sufficiently large n

$$\alpha_n=\alpha_n^0+\frac{\kappa_n}{n}, \qquad \{\kappa_n\}\in\ell_2, \tag{25}$$

wherein α_n^0 is the weight number of L_0 and is equal to $\alpha_n^0=\frac{1}{2}(H-h)$.

Proof. By taking (18), we have

$$\begin{aligned}\dot{\Delta}(\rho)=\dot{\Delta}_0(\rho)-\frac{1}{\rho}\int_{-H+h}^{H-h}&(s-h)N(H,s)sin\rho(s-h)ds\\ &-\frac{1}{\rho^2}\int_{-H+h}^{H-h}N(H,s)cos\rho(s-h)ds.\end{aligned} \tag{26}$$

On the other hand, we know that for a sequence $\{\beta_n\}$,

$$\alpha_n\beta_n=-\dot{\Delta}(\rho_n), \quad \beta_n\neq 0 \tag{27}$$

(see Freiling and Yurko (2001)).

Since $\dot{\Delta}_0(\rho_n^0+\epsilon_n)=\dot{\Delta}_0(\rho_n^0)+\ddot{\Delta}_0(\rho_n^0)\epsilon_n$, by using (26) and (27), we infer that

$$\begin{aligned}\alpha_n\beta_n &= \alpha_n^0\beta_n^0-\ddot{\Delta}_0(\rho_n^0)\epsilon_n\\ &\quad+\frac{1}{\rho_n}\int_{-H+h}^{H-h}(s-h)N(H,s)sin\rho_n(s-h)ds\\ &\quad+\frac{1}{\rho_n^2}\int_{-H+h}^{H-h}N(H,s)cos\rho_n(s-h)ds,\end{aligned}$$

and consequently $\alpha_n=\alpha_n^0+\frac{\kappa_n}{n}$, where

$$\begin{aligned}\kappa_n &= -n(H-h)^2cos\frac{(2n+1)\pi}{2}O(1)\\ &\quad+\frac{2(H-h)}{\pi}O(1)\int_{-H+h}^{H-h}(s-h)N(H,s)sin\rho_n(s-h)ds\\ &\quad+\frac{2(H-h)}{(2n+1)\pi}O(1)\int_{-H+h}^{H-h}N(H,s)cos\rho_n(s-h)ds.\end{aligned}$$

The proof is completed.

Inverse Problem 1. Given the Weyl function $M(\rho)$, construct the potential $q(x)$ and the coefficient β.

3. The Uniqueness Theorem

This section contains the main theorem which clarifies the aim of this paper. For this reason together with $L=L(q(x),\beta)$, we can also consider a boundary value problem $\widetilde{L}=L(\widetilde{q}(x),\widetilde{\beta})$. If a certain symbol e denotes an object related to L, then the corresponding symbol $\widetilde{e}$ with tilde denotes the analogous object related to $\widetilde{L}$, and $\widehat{e}:=e-\widetilde{e}$.

Theorem 4. If $M(\rho)=\widetilde{M}(\rho)$ then $q(x)=\widetilde{q}(x)$ and $\beta=\widetilde{\beta}$. Thus the specification of the Weyl function uniquely determines the BVP(L).

Proof. We consider the matrix $P(x,\rho)=(P_{j,k}(x,\rho))_{j,k=1,2}$ by the formula

$$P(x,\rho)\begin{pmatrix}\widetilde{\varphi_1}(x,\rho) & \widetilde{\phi_1}(x,\rho)\\ \widetilde{\varphi_2}(x,\rho) & \widetilde{\phi_2}(x,\rho)\end{pmatrix}=\begin{pmatrix}\varphi_1(x,\rho) & \phi_1(x,\rho)\\ \varphi_2(x,\rho) & \phi_2(x,\rho)\end{pmatrix}. \qquad (28)$$

So

$$\begin{cases} \varphi_1(x,\rho) = P_{11}(x,\rho)\widetilde{\varphi_1}(x,\rho) + P_{12}(x,\rho)\widetilde{\varphi_2}(x,\rho), \\ \\ \phi_1(x,\rho) = P_{11}(x,\rho)\widetilde{\phi_1}(x,\rho) + P_{12}(x,\rho)\widetilde{\phi_2}(x,\rho). \end{cases} \tag{29}$$

and by virtue of (22), we have

$$\begin{cases} P_{11}(x,\rho) = \varphi_1(x,\rho)\widetilde{\phi_2}(x,\rho) - \phi_1(x,\rho)\widetilde{\varphi_2}(x,\rho), \\ \\ P_{12}(x,\rho) = \phi_1(x,\rho)\widetilde{\varphi_1}(x,\rho) - \varphi_1(x,\rho)\widetilde{\phi_1}(x,\rho). \end{cases} \tag{30}$$

Using (23) and (30), we calculate

$$\begin{cases} P_{11}(x,\rho) = \varphi_1(x,\rho)\widetilde{C_2}(x,\rho) - C_1(x,\rho)\widetilde{\varphi_2}(x,\rho) \\ +\widehat{M}(\rho)\varphi_1(x,\rho)\widetilde{\varphi_2}(x,\rho), \\ \\ P_{12}(x,\rho) = C_1(x,\rho)\widetilde{\varphi_1}(x,\rho) - \varphi_1(x,\rho)\widetilde{C_1}(x,\rho) \\ -\widehat{M}(\rho)\varphi_1(x,\rho)\widetilde{\varphi_1}(x,\rho). \end{cases}$$

On the other hand, from the hypothesis $M(\rho) = \widetilde{M}(\rho)$, we get

$$\begin{cases} P_{11}(x,\rho) = \varphi_1(x,\rho)\widetilde{C_2}(x,\rho) - C_1(x,\rho)\widetilde{\varphi_2}(x,\rho), \\ \\ P_{12}(x,\rho) = C_1(x,\rho)\widetilde{\varphi_1}(x,\rho) - \varphi_1(x,\rho)\widetilde{C_1}(x,\rho), \end{cases}$$

and consequently $P_{1k}(x,\rho),\ k = 1,2$ are entire in ρ for each fixed x. It follows from (21) and (30) that

$$\begin{cases} P_{11}(x,\rho) = 1 + (\varphi_1(x,\rho) - \widetilde{\varphi_1}(x,\rho))\, \frac{\widetilde{\psi_2}(x,\rho)}{\widetilde{\Delta_1}(\rho)} \\ -\left(\frac{\psi_1(x,\rho)}{\Delta_1(\rho)} - \frac{\widetilde{\psi_1}(x,\rho)}{\widetilde{\Delta_1}(\rho)}\right)\widetilde{\varphi_2}(x,\rho), \\ \\ P_{12}(x,\rho) = \left(\frac{\psi_1(x,\rho)}{\Delta_1(\rho)} - \frac{\widetilde{\psi_1}(x,\rho)}{\widetilde{\Delta_1}(\rho)}\right)\varphi_1(x,\rho) \\ -(\varphi_1(x,\rho) - \widetilde{\varphi_1}(x,\rho))\,\frac{\psi_1(x,\rho)}{\Delta_1(\rho)}. \end{cases} \tag{31}$$

Let $G_\delta = \{\rho; |\rho - \rho_n| \geq \delta,\ n = 0, \pm 1, \pm 2, ...\}$ where δ is a sufficiently small number. Similar to those presented in Neamaty and Khalili (2013), it follows from (16), (17) and (19) that for $v = 0, 1$,

$$|\varphi^{(v)}(x,\rho)| \leq C|\rho|^v exp(|Im\rho|(x-h)), \tag{32}$$

$$|\Delta(\rho)| \geq C_\delta exp(|Im\rho|(H-h)), \quad \rho \in G_\delta, \tag{33}$$

$$|\psi^{(v)}(x,\rho)| \leq C|\rho|^v exp(|Im\rho|(H-h-x)). \tag{34}$$

Taking (21), (33) and (34), we have for $\rho \in G_\delta$,

$$|\phi^{(v)}(x,\rho)| \leq C_\delta |\rho|^{v-2} exp(-|Im\rho|x). \tag{35}$$

Denote $G_\delta^0 = G_\delta \cap \widetilde{G_\delta}$. By virtue of (32) and (35), we get as $\rho \in G_\delta^0$, and sufficiently large ρ

$$(\varphi_1(x,\rho) - \widetilde{\varphi_1}(x,\rho)) \frac{\widetilde{\psi_2}(x,\rho)}{\widetilde{\Delta_1}(\rho)} \simeq 0,$$

$$\left(\frac{\psi_1(x,\rho)}{\Delta_1(\rho)} - \frac{\widetilde{\psi_1}(x,\rho)}{\widetilde{\Delta_1}(\rho)} \right) \widetilde{\varphi_2}(x,\rho) \simeq 0,$$

$$\left(\frac{\psi_1(x,\rho)}{\Delta_1(\rho)} - \frac{\widetilde{\psi_1}(x,\rho)}{\widetilde{\Delta_1}(\rho)} \right) \varphi_1(x,\rho) \simeq 0,$$

$$(\varphi_1(x,\rho) - \widetilde{\varphi_1}(x,\rho)) \frac{\psi_1(x,\rho)}{\Delta_1(\rho)} \simeq 0.$$

Therefore by using (31), we infer $P_{11}(x,\rho) = 1$ and $P_{12}(x,\rho) = 0$. Now together with (29), this yields $\widetilde{\varphi}(x,\rho) = \varphi(x,\rho)$ and $\widetilde{\phi}(x,\rho) = \phi(x,\rho)$ for all $x, \ \rho$. Thus $q(x) = \widetilde{q}(x)$ for all x and $\beta = \widetilde{\beta}$. Theorem 4 is proved.

Corollary 1. If $\rho_n = \widetilde{\rho}_n$ and $\alpha_n = \widetilde{\alpha}_n$ then $L = \widetilde{L}$. Thus, the specification of the spectral data $\{\rho_n, \alpha_n\}$ uniquely determines the BVP(L).

Proof. See Theorem 4 in Topsakal and Amirov (2010).

4. An Example

Now in this section we will give an application of a singular Sturm-Liouville equation in applied problems. In the sequel the approximate solution is obtained by using the variational iteration method (VIM). The computations of this problem have been performed on a PC taking some programs written in Mathematica.

Example 1. The problem of the seismic response of earth dams is a boundary value problem wherein we can survey a displacement of dams in the case that a special stress is exerted to those (see Neamaty and Khalili (2014)).

The partial differential equation for this problem with the stress $\tau_{yx}(x;t) =$

$G\left(\frac{\partial u}{\partial x}+\frac{u}{x-H}\right)$ can be written in the following form

$$\frac{1}{x}\frac{\partial}{\partial x}\left(x\left(\frac{\partial u}{\partial x}+\frac{u}{x-H}\right)\right)=C_b^2\frac{\partial^2 u}{\partial t^2},$$

where $\rho_s/G_b = C_b^2$ for the average shear modulus of the soil $G = G_b$, and the mass density of the soil ρ_s. Then this equation reduces to the singular Sturm-Liouville equation in the form

$$-y''+\frac{q_0(x)}{(x-H)^2}y=\lambda y,\quad h<x<H, \tag{36}$$

where $q_0(x)=\frac{4Hx-H^2}{4x^2}$. We can rewrite the equation as

$$-y''+\left(\frac{\frac{3}{4}}{(x-H)^2}+\frac{\frac{-1}{2H}}{x-H}+q(x)\right)y=\lambda y,\quad h<x<H, \tag{37}$$

where $h,\ H$ are positive real numbers and $q(x)\in L_2(h,H)$.

Considering the following boundary conditions

$$U(y):=y'(h)-\beta y(h)=0,\qquad V(y):=y(H)=0, \tag{38}$$

we have a boundary value problem for a such applied problem. Now for solving the problem (36) and (38) by the VIM, we reduce it to the form

$$-y''+\frac{\nu}{(x-H)^2}y=\lambda y,\quad h\le x<H, \tag{39}$$

for a real parameter ν. In the case $\nu=0$, this equation has the solution

$$\begin{aligned} y_0(x)=&\frac{\sqrt{\lambda}cos\sqrt{\lambda}h-\beta sin\sqrt{\lambda}h}{\sqrt{\lambda}}cos\sqrt{\lambda}x\\ &+\frac{\beta cos\sqrt{\lambda}h+\sqrt{\lambda}sin\sqrt{\lambda}h}{\sqrt{\lambda}}sin\sqrt{\lambda}x,\end{aligned}$$

with initial conditions $y(h)=1$ and $y'(h)=\beta$. To find the approximate solution by using the VIM, we have the following correction functional

$$\begin{aligned} y_{n+1}(x)=&\,y_n(x)\\ &+\int_h^x\mu\left\{\frac{d^2y_n(s)}{ds^2}-\frac{\nu}{(s-H)^2}\widetilde{y}_n(s)+\lambda y_n(s)\right\}ds, \end{aligned}\tag{40}$$

where $\widetilde{y}_n$ is considered as a restricted variation. In the following, by making the functional stationary

$$\begin{aligned}\delta y_{n+1}(x) &= (1-\mu'(x))\delta y_n(x) + \mu(x)\delta y_n'(x) \\ &\quad + \int_h^x \left\{\frac{d^2\mu(s)}{ds^2} + \lambda\mu(s)\right\}\delta y_n(s)\,ds,\end{aligned}$$

we have the stationary conditions

$$\frac{d^2\mu(s)}{ds^2} + \lambda\mu(s) = 0, \quad \mu(x) = 0, \quad \mu'(x) = 1.$$

We give the Lagrange multiplier $\mu = \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}(x-s)\right)$. So we take the iteration formula

$$\begin{aligned}y_{n+1}(x) &= y_n(x) + \int_h^x \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}(x-s)\right) \\ &\quad \times \left\{y_n''(s) - \frac{\nu}{(s-H)^2}y_n(s) + \lambda y_n(s)\right\}ds. \qquad (41)\end{aligned}$$

Let us start with an initial approximation $y_0(x)$. Substituting y_0 in (41), we have

$$\begin{aligned}y_1(x) \quad = \quad & \frac{\sqrt{\lambda}cos\sqrt{\lambda}h - \beta sin\sqrt{\lambda}h}{\sqrt{\lambda}}cos\sqrt{\lambda}x \\ & + \frac{\beta cos\sqrt{\lambda}h + \sqrt{\lambda}sin\sqrt{\lambda}h}{\sqrt{\lambda}}sin\sqrt{\lambda}x \\ & - \int_h^x \frac{\nu\sin\left(\sqrt{\lambda}(x-s)\right)}{\sqrt{\lambda}(s-H)^2} \\ & \times\left\{\frac{\sqrt{\lambda}cos\sqrt{\lambda}h - \beta sin\sqrt{\lambda}h}{\sqrt{\lambda}}cos\sqrt{\lambda}s\right. \\ & \quad \left. + \frac{\beta cos\sqrt{\lambda}h + \sqrt{\lambda}sin\sqrt{\lambda}h}{\sqrt{\lambda}}sin\sqrt{\lambda}s\right\}ds.\end{aligned}$$

By taking Mathematica, we can obtain the following approximate solution

$$
\begin{aligned}
y(x) \quad \approx \quad & y_1(x) = \frac{\sqrt{\lambda}cos\sqrt{\lambda}h - \beta sin\sqrt{\lambda}h}{\sqrt{\lambda}} cos\sqrt{\lambda}x \\
& + \frac{\beta cos\sqrt{\lambda}h + \sqrt{\lambda}sin\sqrt{\lambda}h}{\sqrt{\lambda}} sin\sqrt{\lambda}x \\
& + \frac{\nu(\sqrt{\lambda}cos\sqrt{\lambda}h - \beta sin\sqrt{\lambda}h)}{2\lambda(h-H)} \\
& \times \Big(sin\sqrt{\lambda}(2h-x) - sin\sqrt{\lambda}x \\
& -2\sqrt{\lambda}(h-H)Ci(2\sqrt{\lambda}(h-H))cos\sqrt{\lambda}(2H-x) \\
& +2\sqrt{\lambda}(h-H)Ci(-2\sqrt{\lambda}(H-x))cos\sqrt{\lambda}(2H-x) \\
& -2\sqrt{\lambda}hSi(2\sqrt{\lambda}(-h+H))sin\sqrt{\lambda}(2H-x) \\
& +2\sqrt{\lambda}HSi(2\sqrt{\lambda}(-h+H))sin\sqrt{\lambda}(2H-x) \\
& +2\sqrt{\lambda}hSi(2\sqrt{\lambda}(H-x))sin\sqrt{\lambda}(2H-x) \\
& -2\sqrt{\lambda}HSi(2\sqrt{\lambda}(H-x))sin\sqrt{\lambda}(2H-x)\Big) \\
& + \frac{\nu(\beta cos\sqrt{\lambda}h + \sqrt{\lambda}sin\sqrt{\lambda}h)}{2\lambda(h-H)} \\
& \times \Big(- cos\sqrt{\lambda}(2h-x) + cos\sqrt{\lambda}x \\
& -2\sqrt{\lambda}(h-H)Ci(2\sqrt{\lambda}(h-H))sin\sqrt{\lambda}(2H-x) \\
& +2\sqrt{\lambda}(h-H)Ci(-2\sqrt{\lambda}(H-x))sin\sqrt{\lambda}(2H-x) \\
& +2\sqrt{\lambda}hSi(2\sqrt{\lambda}(-h+H))cos\sqrt{\lambda}(2H-x) \\
& -2\sqrt{\lambda}HSi(2\sqrt{\lambda}(-h+H))cos\sqrt{\lambda}(2H-x) \\
& -2\sqrt{\lambda}hSi(2\sqrt{\lambda}(H-x))cos\sqrt{\lambda}(2H-x) \\
& +2\sqrt{\lambda}HSi(2\sqrt{\lambda}(H-x))cos\sqrt{\lambda}(2H-x)\Big).
\end{aligned}
$$

The approximate solution is shown in Table 1 and Figure 1 for $h = 10$, $H = 100$, $\lambda = 4$, $\beta = 5$ and $\nu = 1$.

Table 1: The Approximate Solution using $\lambda = 4$

x	$y(x)$
10	1
10.1	1.040940331
10.2	1.039023678
10.3	0.9611099142
10.4	0.8751318014
10.5	0.750000000
10.6	0.600000000
10.7	0.4206852847

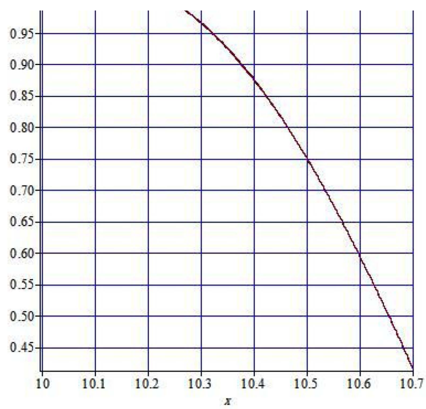

Figure 1: The approximate solution using $\lambda = 4$.

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